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# Holomorphic Jackson's theorems in polydiscs

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**Abstract**

The purpose of this article is to establish Jackson-type inequality in the polydiscs $U^N$ of $\mathbb{C}^N$ for holomorphic spaces X, such as Bergman-type spaces, Hardy spaces, polydisc algebra and Lipschitz spaces. Namely,

$$E_{\vec{k}}(f, \mathrm{X}) \lesssim \omega\left(\overrightarrow{1/k}, f, \mathrm{X}\right),$$

where $E_{\vec{k}}(f, \mathrm{X})$ is the deviation of the best approximation of $f \in \mathrm{X}$ by polynomials of degree at most $k_j$ about the $j$th variable $z_j$ with respect to the X-metric and $\omega(\overrightarrow{1/k}, f, \mathrm{X})$ is the corresponding modulus of continuity.



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## 1. Introduction

Jackson's theorem (see [4]) is an important result in the theory of approximation treating the deviation of the best approximation of a function by polynomials. It has been

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established for various classes of functions and for moduli of continuity of arbitrary order (see [4,17,9,15,2,6,7,14,16]). In this paper we are concerned with an extension of Jackson's theorem to holomorphic function spaces in the unit polydiscs. This is motivated by the following well-known versions of Jackson's theorem in the complex unit disc due to Storozhenko [12,13], Sewell [11], and Jackson [8], respectively.

**Theorem 1.1** (*Storozhenko [12,13]*). *Let $f \in H^p(U)$, the Hardy space in the complex unit disc $U$ with $0 < p < \infty$, then for any $k \in \mathbb{N}$ we have*

$$\inf_{P_k \in \mathrm{M}_k} \|f - P_k\|_{H^p(U)} \leqslant C(p) \sup_{|h| \leqslant 1/k} \sup_{z \in U} |f(e^{ih}z) - f(z)|,$$

*where $\mathrm{M}_k$ denotes the set of all polynomials of degree at most $k$.*

**Theorem 1.2** (*Sewell [11]*). *Let $f \in A(U)$, the disc algebra, then*

$$\inf_{P_k \in \mathrm{M}_k} \max_{z \in U} |f(z) - P_k(z)| \leqslant C \sup_{|h| \leqslant 1/k} \sup_{z \in \partial U} |f(e^{ih}z) - f(z)|.$$

**Theorem 1.3** (*Jackson [8]*). *Let $f \in Lip_\gamma(U)$, the Lipschitz space, then*

$$\inf_{P_k \in \mathrm{M}_k} \max_{z \in U} |f(z) - P_k(z)| = O\left(\frac{1}{k^\gamma}\right).$$

Theorem 1.1 has been extended by Colzani [3] to the unit polydisc with the polynomial spaces $\mathrm{M}_k$ replaced by $\mathrm{M}_k(U^N)$, the polynomial spaces of degree $k \in \mathbb{N}$ in $U^N$. In [1], Anderson et al. considered the generalization of Theorem 1.3 to Jordan domains of the complex plane.

The main purpose of this article is to extend Theorems 1.2 and 1.3 to the case of polydiscs. Moreover, many other holomorphic spaces are also considered, such as Bergman-type spaces, Hardy spaces and Sobolev spaces. Instead of the polynomial approximation in $\mathrm{M}_k(U^N)$ considered by Colzani, we shall consider further the approximation in $\mathrm{W}_{\vec{k}}(U^N)$ of vector degree, i.e., the set of all polynomials whose degree about the $j$th variable's are at most $k_j$ for each $j = 1, \ldots, N$. Here the degree $\vec{k}$ is a vector $\vec{k} = (k_1, \ldots, k_N)$.

## 2. Preliminaries

Let $U^N$ be the unit polydisc of $\mathbb{C}^N$ with the Shilov boundary $T^N$ (see [10]). For any $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, if we denote

$$\|z\|_{\max} = \max_{j=1,\ldots,N} |z_j|,$$

then $U^N$ is the unit ball with respect to the above norm.

We shall use the following abbreviated notations. We write

$$\vec{k}^m = \prod_{j=1}^{N} k_j^m,$$

$$\overrightarrow{1/k} = \left(\frac{1}{k_1}, \ldots, \frac{1}{k_N}\right),$$

$$\overrightarrow{\lambda\delta} = (\lambda_1\delta_1, \ldots, \lambda_N\delta_N)$$

for any $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$, $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\delta = (\delta_1, \ldots, \delta_N)$, and non-negative integer $m$. We also write $d\varphi = d\varphi_1 \cdots d\varphi_N$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, and by $\alpha > 0$ we mean that each component $\alpha_j > 0$. In polar coordinates we write $z = r\zeta$ with $r \in I^N \stackrel{\text{def}}{=} [0, 1)^N$ and $\zeta \in T^N$.

For any non-negative functions $f$ and $g$, we write $f \lesssim g$, if there exists a positive constant $C$ such that $f \leqslant Cg$. Analogously, we also write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

To consider the holomorphic Jackson's theorem in polydiscs, our approach is to construct the best approximation of polynomials with a kind of complex measures on $T^N$, whose total variations are given by *generalized Jackson's kernels*:

$$T_k^\beta(\theta) \stackrel{\text{def}}{=} \left(\frac{\sin\dfrac{k\theta}{2}}{\sin\dfrac{\theta}{2}}\right)^{2\beta}.$$

For generalized Jackson kernels, we need the following results:

**Lemma 2.1** (*See DeVore et al. [4, p. 203]*). *Let $k, \beta \in \mathbb{N}$.*

(A) *There exists constants $C_{l,\beta}(k)$, such that the generalized Jackson Kernel*

$$T_k^\beta(\theta) = \sum_{l=0}^{\beta(k-1)} C_{l,\beta}(k) \cos l\theta.$$

(B) *For each $\beta \in \mathbb{N}$,*

$$B_{k,\beta} \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} T_k^\beta(\theta)\, d\theta \sim k^{2\beta-1}$$

*as $k \to +\infty$; and there is a constant $C_\beta$ for which*

$$\frac{1}{B_{k,\beta}} \int_0^\pi \theta^u T_k^\beta(\theta)\, d\theta \leqslant C_\beta k^{-u}, \quad u = 0, 1, \ldots, 2\beta - 2.$$

**Lemma 2.2.** *Let $\beta > 1$ and $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$, then*

$$\int_{[-\pi,\pi)^N} \prod_{j=1}^{N} \left|T_{k_j}^\beta(\varphi_j)\right| (\|\varphi k\|_{\max} + 1)\, d\varphi = O(\vec{k}^{2\beta-1}).$$

**Proof.** For any $x \in [0, \frac{\pi}{2}]$, we have $|\sin k_j x| \leqslant k_j |\sin x|$ and $\sin x \geqslant \frac{2}{\pi} x$. Denote $t = 2\beta$, then

$$\begin{aligned}
&\int_{[-\pi,\pi)} \left| \frac{\sin \frac{k_j}{2}\varphi_j}{\sin \frac{\varphi_j}{2}} \right|^t \left(|\varphi_j k_j| + 1\right) d\varphi_j \\
&= 2\int_{[0,\frac{\pi}{k_j})} \left| \frac{\sin \frac{k_j}{2}\varphi_j}{\sin \frac{\varphi_j}{2}} \right|^t \left(\varphi_j k_j + 1\right) d\varphi_j + 2\int_{[\frac{\pi}{k_j},\pi)} \left| \frac{\sin \frac{k_j}{2}\varphi_j}{\sin \frac{\varphi_j}{2}} \right|^t \left(\varphi_j k_j + 1\right) d\varphi_j \\
&\leqslant 2\int_{[0,\frac{\pi}{k_j})} k_j^t \left(\varphi_j k_j + 1\right) d\varphi_j + 2\pi^t \int_{[\frac{\pi}{k_j},\pi)} \varphi_j^{-t} \left(\varphi_j k_j + 1\right) d\varphi_j \\
&= 2\left(\frac{\pi^2}{2} k_j^{t-1} + \pi k_j^{t-1}\right) + 2\pi^t \left[\frac{k_j}{t-2}(k_j^{t-2} - 1)\pi^{2-t} + \frac{1}{t-1}(k_j^{t-1} - 1)\pi^{1-t}\right] \\
&= O(k_j^{t-1}).
\end{aligned}$$

Since

$$\|\varphi k\|_{\max} + 1 = |\varphi_{j_0} k_{j_0}| + 1 \leqslant \prod_{j=1}^{N} (|\varphi_j k_j| + 1)$$

for some $j_0$, we have

$$\begin{aligned}
&\int_{[-\pi,\pi)^N} \prod_{j=1}^{N} \left| \frac{\sin \frac{k_j}{2}\varphi_j}{\sin \frac{\varphi_j}{2}} \right|^t (\|\varphi k\|_{\max} + 1)\, d\varphi \\
&\leqslant \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j=1}^{N} \left| \frac{\sin \frac{k_j \varphi_j}{2}}{\sin \frac{\varphi_j}{2}} \right|^t \prod_{j=1}^{N} (|\varphi_j k_j| + 1)\, d\varphi_1 \cdots d\varphi_N \\
&= \prod_{j=1}^{N} \int_{[-\pi,\pi)} \left| \frac{\sin \frac{k_j \varphi_j}{2}}{\sin \frac{\varphi_j}{2}} \right|^t (|\varphi_j k_j| + 1)\, d\varphi_j = O(\vec{k}^{t-1}).
\end{aligned}$$

This completes the proof. □

Now we introduce the required normalized complex measure $d\mu_k^\rho(\varphi)$ on $[-\pi, \pi)^N$. We fix $a > 0$ and define

$$d\mu_k^\rho(\varphi) = i^N C_N^{\vec{k}} \prod_{j=1}^{N} (\rho_j e^{i\varphi_j})^{1-k_j} \left[ \frac{1 - (\rho_j e^{i\varphi_j})^{k_j+1}}{1 - \rho_j e^{i\varphi_j}} \right]^{a+1} d\varphi,$$

where

$$C_N^{\vec{k}} = \prod_{j=1}^{N} (2\pi i A_{k_j}^a)^{-1},$$

$$A_{k_j}^a = \frac{\Gamma(k_j + a)}{\Gamma(k_j)\Gamma(a+1)}.$$

Then $d\mu_k^\rho(\varphi)$ are normalized measures on $[-\pi, \pi)^N$ for any $\rho \in (0, 1]^N$ and $k \in \mathbb{N}^N$.

There are two important facts related to these measures. One is that the total variation of $d\upsilon_k \overset{\text{def}}{=} d\mu_k^\rho(\varphi)$ for $\rho = (1, \dots, 1)$ is given by the generalized Jackson kernels; in fact, by direct calculation we have

$$d|\upsilon_k| = |C_N^{\vec{k}}| \prod_{j=1}^{N} T_{k_j}^{\frac{a+1}{2}}(\varphi_j)\, d\varphi.$$

The other is that the associated operators $P_k$

$$P_k[f](z) = \int_{[-\pi,\pi)^N} f(\rho e^{i\varphi} z)\, d\mu_k^\rho(\varphi), \quad z \in U^N, \tag{2.1}$$

will provide the best approximation of polynomials.

**Lemma 2.3.** *Let $f$ be holomorphic on $U^N$ and $\rho \in (0, 1]^N$.*

(A) *$P_k[f](z)$ is a polynomial of degree at most $|k| - N$; moreover its jth variable's degree is at most $k_j - 1$, $j = 1, \dots, N$; $P_k[1] \equiv 1$.*

(B) *$P_k[f](z)$ has another integral formula*:

$$P_k[f](z) = C_N^{\vec{k}} \int_{\rho T^N} f(\lambda z) \prod_{j=1}^{N} \lambda_j^{-k_j} (1 - \lambda_j)^{-(a+1)}\, d\lambda \tag{2.2}$$

*for any $\rho \in (0, 1)^N$.*

**Proof.** We first prove assertion (B). For any $\rho = (\rho_1, \dots, \rho_N) \in (0, 1)^N$, we can rewrite (2.1) as

$$P_k[f](z) = C_N^{\vec{k}} \int_{\rho T^N} f(\lambda z) \prod_{j=1}^{N} \lambda_j^{-k_j} \left( \frac{1 - \lambda_j^{k_j+1}}{1 - \lambda_j} \right)^{a+1} d\lambda. \tag{2.3}$$

Notice that the item in the brackets is a polynomial of $\lambda$ for $k = (k_1, \dots, k_N) \in \mathbb{N}^N$, so that the integrand is holomorphic in $\{\lambda \in U^N : \lambda_j \neq 0,\ j = 1, \dots, N\}$.

From the binomial series $(1 - \omega)^{a+1} = 1 + \sum_{l=1}^{\infty} b_l w^l$ for any $|w| < 1$, we have

$$(1 - \lambda_j^{k_j+1})^{a+1} = 1 + \sum_{l=1}^{\infty} b_l \lambda_j^{(k_j+1)l}, \quad |\lambda_j| < 1,$$

so that the integrand in (2.3) can be split into two parts

$$f(\lambda z) \prod_{j=1}^{N} \lambda_j^{-k_j} \left( \frac{1 - \lambda_j^{k_j+1}}{1 - \lambda_j} \right)^{a+1}$$
$$= f(\lambda z) \prod_{j=1}^{N} \lambda_j^{-k_j} (1 - \lambda_j)^{-(a+1)} + \sum_{l=1}^{\infty} f(\lambda z) \prod_{j=1}^{N} \lambda_j^{-k_j} (1 - \lambda_j)^{-(a+1)} b_l \lambda_j^{(k_j+1)l}.$$

Since the second term is holomorphic in $U^N$, its integral over $\rho T^N$ vanishes, so that the first item produces formula (2.2).

To prove (A), write $z \in \mathbb{C}^N$ in the form $z = (z', z_N)$ where $z' \in \mathbb{C}^{N-1}$. Now we apply formula (2.2), write the integral over $\rho' T^{N-1} \times \rho_N T$ in iterated form, and then apply the residue theorem to the integral over $\rho_N T$ to obtain

$$P_k[f](z) = C \int_{\rho' T^{N-1}} \prod_{j=1}^{N-1} \lambda_j^{-k_j} (1-\lambda_j)^{-(a+1)} \operatorname{Res}\big(g(\lambda_N), 0\big)\, d\lambda', \tag{2.4}$$

where

$$g(\lambda_N) = f(\lambda z)\lambda_N^{-k_N}(1-\lambda_N)^{-(a+1)}.$$

From the multiple power series $f(z) = \sum_{\beta \geqslant 0}^{\infty} b_\beta z^\beta$ and the binomial series

$$(1-\lambda_j)^{-(a+1)} = \sum_{l=0}^{\infty} \frac{\Gamma(l+a+1)}{l!\Gamma(a+1)} \lambda_j^l$$

we have

$$\begin{aligned} g(\lambda_N) &= f(\lambda z)\lambda_N^{-k_N}(1-\lambda_N)^{-(a+1)} \\ &= \sum_{\beta_1,\ldots,\beta_N=0}^{\infty} b_\beta (\lambda_1 z_1)^{\beta_1} \cdots (\lambda_N z_N)^{\beta_N} \lambda_N^{-k_N} \sum_{l=0}^{\infty} \frac{\Gamma(l+a+1)}{l!\Gamma(a+1)} \lambda_N^l \\ &= \sum_{\beta_1,\ldots,\beta_N,l=0}^{\infty} b_\beta (\lambda_1 z_1)^{\beta_1} \cdots (\lambda_{N-1} z_{N-1})^{\beta_{N-1}} z_N^{\beta_N} \frac{\Gamma(l+a+1)}{l!\Gamma(a+1)} \lambda_N^{\beta_N - k_N + l}. \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Res}\big(g(\lambda_N), 0\big) \\ &\quad = \sum_{\beta_N + l = k_N - 1} b_\beta (\lambda_1 z_1)^{\beta_1} \cdots (\lambda_{N-1} z_{N-1})^{\beta_{N-1}} z_N^{\beta_N} \frac{\Gamma(l+a+1)}{l!\Gamma(a+1)}, \end{aligned}$$

which is a polynomial of degree at most $k_N - 1$ with respect to the variable $z_N$.

From (2.4), we know that $P_k[f](z)$ is a polynomial of $z_N$ with degree at most $k_N - 1$. In view of the symmetry of the variables $z_j$, the proof is complete. □

In [3], Colzani considered the approximation of polynomials from the space $\mathrm{M}_k \stackrel{\text{def}}{=} \mathrm{M}_k(U^N)$ with scalar degree in $U^N$. In this article we shall consider the best approximation by polynomials with vector degrees. More precisely, for any semi-normed space X on $U^N$ with semi-norm $\|\cdot\|_{\mathrm{X}}$, we define the *best approximation of f of order $\vec{k}$ by* X as

$$E_{\vec{k}}(f, \mathrm{X}) \stackrel{\text{def}}{=} \inf_{Q_k \in \mathrm{W}_{\vec{k}}} \|f - Q_k\|_{\mathrm{X}},$$

where $\mathrm{W}_{\vec{k}} \stackrel{\text{def}}{=} \mathrm{W}_{\vec{k}}(U^N)$ is the polynomial space of vector degrees, i.e., the set of all polynomials whose degree about the *j*th variable's are at most $k_j$ for each $j = 1, \ldots, N$.

The degree of approximation is provided by the moduli of continuity and the moduli of smoothness.

**Definition 2.4.** Let X be a semi-normed space on $U^N$ with semi-norm $\|\cdot\|_X$. For any $f \in X$, and $\delta = (\delta_1, \ldots, \delta_N) > 0$, we define the modulus of continuity of *f* as

$$\omega(\vec{\delta}, f, X) \stackrel{\text{def}}{=} \sup_{\overrightarrow{|\theta' - \theta''|} \in I_{\delta,N}} \left\| f(e^{i\theta'} z) - f(e^{i\theta''} z) \right\|_X,$$

where $\overrightarrow{|\theta' - \theta''|} = (|\theta'_1 - \theta''_2|, \ldots, |\theta'_N - \theta''_N|)$, $I_{\delta,N} = [0, \delta_1) \times \cdots \times [0, \delta_N)$ and $e^{i\theta'} z = (e^{i\theta'_1} z_1, \ldots, e^{i\theta'_n} z_n)$.

**Definition 2.5.** Let X be a semi-normed space on $U^N$. For any $f \in X$, $l \in \mathbb{N}$, and $\delta \in [0, \infty)$, we define the slice modulus of smoothness of order *l* on X of *f* as

$$\omega_l(\delta, f, X) \stackrel{\text{def}}{=} \sup_{0 \leqslant h \leqslant \delta} \sup_{\zeta \in U^N} \left\| \Delta_h^l f(\zeta) \right\|_X,$$

where the higher differences are defined as

$$\Delta_h^l f(\zeta) = \Delta_h \Delta_h^{l-1} f(\zeta) = \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} f(e^{imh} \zeta).$$

To study Jackson's theorem on $U^N$, we need the following key lemma:

**Lemma 2.6.** *Let* X *be a semi-normed space on* $U^N$ *and* $f \in X$, *then for any* $k \in \mathbb{N}^N$

$$E_{\vec{k}}(f, X) \lesssim \omega(\overrightarrow{1/k}, f, X)$$

*provided the following two conditions hold*:

(A) *There exists* $s > 0$ *such that* $\omega^s(\overrightarrow{\lambda\delta}, f, X) \leqslant (\|\lambda\|_{\max} + 1)\omega^s(\vec{\delta}, f, X)$ *for any* $\lambda, \delta \in [0, \infty)^N$.

(B) *For any* $m \in \mathbb{N}^N$, *there exists polynomial* $G_m$ *of degree at most* $s_1(m_j - s_2)$ *with respect to the jth variable, where* $s_1$ *and* $s_2$ *are two absolute constants in* $\mathbb{N}$, *such that* $\|G_m - f\|_X \lesssim \omega(\overrightarrow{1/m}, f, X)$.

**Proof.** Fix $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$. We take $m_j = [k_j/s_1] + s_2$ then $s_1(m_j - s_2) \leqslant k_j$, so that there exists a polynomial $G_m$ of degree at most $s_1(m_j - s_2) \leqslant k_j$ with respect to the *j*th variable. By assumption, we get

$$\begin{aligned} \|G_m[f] - f\|_X^s &\lesssim \omega^s\left(\overrightarrow{1/m}, f, X\right) \\ &\lesssim \left(\left\|\frac{k}{m}\right\|_{\max} + 1\right) \omega^s\left(\overrightarrow{1/k}, f, X\right) \lesssim \omega^s\left(\overrightarrow{1/k}, f, X\right), \end{aligned}$$

where $k/m = (k_1/m_1, \ldots, k_N/m_N)$.

Since $E_{\vec{k}}(f, \mathrm{X}) = \inf_{\mathcal{Q}_k \in \mathrm{W}_{\vec{k}}} \|f - \mathcal{Q}_k\|_{\mathrm{X}}$, it follows that

$$E_{\vec{k}}(f, \mathrm{X}) \leqslant \|G_m[f] - f\|_{\mathrm{X}} .$$

Combining the above results, we have

$$E_{\vec{k}}(f, \mathrm{X}) \lesssim \omega \left( \overrightarrow{1/k}, f, \mathrm{X} \right),$$

which completes the proof. □

The preceding lemma concerns the approximation of polynomials with vector degrees. A similar result also holds in the scalar case, in which the modulus of smoothness with higher orders is involved.

**Lemma 2.7.** *Let* X *be a semi-normed space,* $f \in \mathrm{X}$ *and* $l \in \mathbb{N}$. *Then for any* $k \in \mathbb{N}$,

$$E_k(f, \mathrm{X}) \lesssim \omega_l(1/k, f, \mathrm{X}),$$

*provided the following two conditions hold*:

(A) *There exists* $s > 0$ *such that* $\omega_l^s(\lambda\delta, f, \mathrm{X}) \leqslant (\lambda + 1)\omega_l^s(\delta, f, \mathrm{X})$ *for any* $\lambda, \delta \in [0, \infty)$.
(B) *there exists a polynomial* $G_n$ *of total degree at most* $s_1(n - s_2)$, *where* $s_1$ *and* $s_2$ *are two absolute constants in* $\mathbb{N}$, *such that* $\|G_n - f\|_{\mathrm{X}} \lesssim \omega_l(1/n, f, \mathrm{X})$.

## 3. Bergman-type spaces

In this section, we consider the polynomial approximation in Bergman-type spaces. The main result of this section is Theorem 3.5.

Let $H(U^N)$ be the set of all holomorphic functions in $U^N$. For any Lebesgue measurable function $f$ in $U^N$, the integral means are defined by

$$M_q(r, f) = \left[ \int_{T^N} |f(r\zeta)|^q \, d\sigma_N(\zeta) \right]^{1/q}, \quad 0 < q < +\infty,$$
$$M_\infty(r, f) = \sup_{\zeta \in T^N} |f(r\zeta)|,$$

where $d\sigma_N(\zeta)$ is the normalized surface measure on $T^N$.

Let $0 < p < \infty$ and $\alpha = (\alpha_1, \dots, \alpha_N) > 0$. We consider the weighted measure

$$dm_\alpha(z) = \prod_{j=1}^{N} (1 - |z_j|^2)^{-1 + p\alpha_j} \, dm(z),$$

where $dm$ is the normalized Lebesgue measure in $U^N$. The weighted Bergman space $L_a^{p,\alpha}(U^N)$ is then defined to be the set of all holomorphic functions $f \in H(U^N)$ with

$$\|f\|_{L_a^{p,\alpha}(U^N)} = \left( \int_{U^N} |f(z)|^p \, dm_\alpha(z) \right)^{1/p} < +\infty.$$

The *Bergman-type space* $H^{p,q,\alpha}(U^N)$ is the set of all $f \in H(U^N)$ for which $\|f\|_{H^{p,q,\alpha}(U^N)} < +\infty$, where

$$\|f\|_{H^{p,q,\alpha}(U^N)} = \left(\int_{I^N} M_q^p(r, f)\, dr_\alpha\right)^{1/p},$$

$$\|f\|_{H^{\infty,q,\alpha}(U^N)} = \sup_{r \in I^N} \prod_{j=1}^{N} (1 - r_j^2)^{\alpha_j} M_q(r, f).$$

Here $dr_\alpha$ denotes the weighted measure on $I^N$

$$dr_\alpha = \prod_{j=1}^{N} (1 - r_j^2)^{-1+p\alpha_j}\, dr.$$

When $p = q$, $H^{p,q,\alpha}(U^N)$ turns out to be the weighted Bergman space $L_a^{p,\alpha}(U^N)$.

Let $P_m[f]$ be the polynomial given by formula (2.1) for any $f \in H(U^N)$. If we consider the following measure on $[-\pi, \pi)^N$:

$$dv_m^{\rho,\eta,a}(\varphi) \overset{\text{def}}{=} |C_N^{\vec{m}}|^\eta \prod_{j=1}^{N} \rho_j^{\eta(1-m_j)} (1 - \rho_j)^{\eta-1} \prod_{j=1}^{N} T_{m_j+1}^{\frac{\eta(a+1)}{2}}(\varphi_j)\, d\varphi \tag{3.1}$$

for any $0 < \rho_j < 1$, $\eta > 0$, and $a > 0$, then we have the following estimates:

**Lemma 3.1.** *For any* $0 < \eta \leqslant 1$, $m = (m_1, \ldots, m_N) \in \mathbb{N}^N$, *and* $f \in H(U^N)$,

$$|P_m[f](z) - f(z)|^\eta \lesssim \int_{[-\pi,\pi)^N} \left|f(e^{i\varphi}z) - f(z)\right|^\eta dv_m^{\rho,\eta,a}(\varphi). \tag{3.2}$$

**Proof.** It is well known that if $g \in H^p(U)$ and $0 < p \leqslant 1$ then for any $0 < r < 1$ (see [5])

$$\left(\int_{-\pi}^{\pi} \left|g(re^{i\theta})\right| d\theta\right)^p \lesssim (1 - r)^{p-1} \int_{-\pi}^{\pi} \left|g(e^{i\theta})\right|^p d\theta. \tag{3.3}$$

By Lemma 2.3, we find that for any $\rho \in (0, 1)^N$

$$P_m[f](z) - f(z) = \int_{[-\pi,\pi)^N} \left[f(\rho e^{i\varphi}z) - f(z)\right] d\mu_m^\rho(\varphi), \tag{3.4}$$

where

$$d\mu_m^\rho(\varphi) = d\mu_{M,N}^\rho(\varphi) \overset{\text{def}}{=} i^N C_N^{\vec{m}} \prod_{j=1}^{N} (\rho_j e^{i\varphi_j})^{1-m_j} \left[\frac{1 - (\rho_j e^{i\varphi_j})^{m_j+1}}{1 - \rho_j e^{i\varphi_j}}\right]^{a+1} d\varphi.$$

Disregarding the constant, we have

$$d\mu_{m,N}^\rho(\varphi) = (\rho_N e^{i\varphi_N})^{1-m_N} \left[\frac{1 - (\rho_N e^{i\varphi_N})^{m_N+1}}{1 - \rho_N e^{i\varphi_N}}\right]^{a+1} d\mu_{m,N-1}^{\rho'}(\varphi')\, d\varphi_N,$$

where $\rho = (\rho', \rho_N)$ and $\varphi = (\varphi', \varphi_N)$.

To deal with the integral over $[-\pi, \pi)^N$ in (3.4), we first rewrite it as an iterated integral and then consider it as the integral over $[-\pi, \pi)$. Therefore,

$$|P_m[f](z) - f(z)| \lesssim \int_{[-\pi,\pi)} |g(\rho e^{i\varphi}, z)|\, d\varphi_N,$$

where $g(\rho e^{i\varphi}, z)$ equals to

$$\left|\int_{[-\pi,\pi)^{N-1}} [f(\rho e^{i\varphi} z) - f(z)]\, d\mu^{\rho'}_{m,N-1}(\varphi')\right| \rho_N^{1-m_N} \left|\frac{1-(\rho_N e^{i\varphi_N})^{m_N+1}}{1-\rho_N e^{i\varphi_N}}\right|^{a+1}.$$

Let $h(\rho e^{i\varphi}, z)$ denote the integral

$$\int_{[-\pi,\pi)^{N-1}} [f(\rho e^{i\varphi} z) - f(z)] \left[\frac{1-(\rho_N e^{i\varphi_N})^{m_N+1}}{1-\rho_N e^{i\varphi_N}}\right]^{a+1} d\mu^{\rho'}_{m,N-1}(\varphi').$$

Then $g(\rho e^{i\varphi}, z) = \rho_N^{1-m_N} |h(\rho e^{i\varphi}, z)|$, so that

$$|P_m[f](z) - f(z)| \lesssim \rho_N^{1-m_N} \int_{[-\pi,\pi)} |h(\rho e^{i\varphi}, z)|\, d\varphi_N.$$

Since $h(\rho e^{i\varphi}, z)$ is a holomorphic function of $\lambda_N = \rho_N e^{i\varphi_N}$ in $A(U)$ for any fixed $z \in U^N$, we can invoke inequality (3.3). The resulting integral over $[-\pi, \pi)^{N-1}$ can be dealt with the same procedure as above. Finally, we can deduce that

$$|P_m[f](z) - f(z)|^\eta \lesssim \int_{[-\pi,\pi)^N} \left|f(e^{i\varphi} z) - f(z)\right|^\eta dv_m^{\rho,\eta,a}(\varphi),$$

as desired. □

**Lemma 3.2.** *Assume* $0 < \eta \leqslant 1$ *and* $\eta(a+1) > 2$. *Then for any* $m \in \mathbb{N}^N$ *and* $\rho_j = 1 - \dfrac{1}{m_j}$ $(j = 1, \ldots, N)$,

$$\int_{[-\pi,\pi)^N} (\|m\varphi\|_{\max} + 1)\, dv_m^{\rho,\eta,a}(\varphi) = O(1) \quad as \quad m \to \infty.$$

**Proof.** Recall that

$$dv_m^{\rho,\eta,a}(\varphi) = |C_N^{\vec{m}}|^\eta \prod_{j=1}^N \rho_j^{\eta(1-m_j)} (1-\rho_j)^{\eta-1} \prod_{j=1}^N T_{m_j+1}^{\frac{\eta(a+1)}{2}}(\varphi_j)\, d\varphi.$$

By Lemma 2.2, we have

$$\int_{[-\pi,\pi)^N} (\|m\varphi\|_{\max} + 1) \prod_{j=1}^N T_{m_j+1}^{\frac{\eta(a+1)}{2}}\, d\varphi = O\left(\vec{m}^{\eta(a+1)-1}\right).$$

Observing that $\prod_{j=1}^{N}(1-\rho_j)^{\eta-1} = \vec{m}^{-(\eta-1)}$, $\rho_j^{\eta(1-m_j)} \sim 1$ and

$$|C_N^{\vec{m}}|^{\eta} \sim \left(\prod_{j=1}^{N} A_{m_j}^{a}\right)^{-\eta} \sim \left(\prod_{j=1}^{N} m_j^{a}\right)^{-\eta} = \vec{m}^{-\eta a},$$

we obtain the desired result. □

Now we can show that Condition (A) in Lemma 2.6 holds for Bergman-type spaces.

**Lemma 3.3.** *Let* $0 < p, q \leqslant \infty$, $s = \min\{1, p, q\}$, $f \in H^{p,q,\alpha}(U^N)$, *then*

$$\omega^s(\overrightarrow{\lambda\delta}, f, H^{p,q,\alpha}(U^N)) \leqslant (\|\lambda\|_{\max}+1)\omega^s(\vec{\delta}, f, H^{p,q,\alpha}(U^N)).$$

**Proof.** By Definition 2.4,

$$\omega(\overrightarrow{\lambda\delta}, f, H^{p,q,\alpha}(U^N)) = \sup_{\overrightarrow{|\theta'-\theta''|}\in I_{\lambda\delta,N}} \left\| f(e^{i\theta'}z) - f(e^{i\theta''}z)\right\|_{H^{p,q,\alpha}(U^N)}.$$

Assume $\overrightarrow{|\theta'-\theta''|} \in I_{\lambda\delta,N}$ and take $m = [\|\lambda\|_{\max}]$. Then

$$|\theta'_j - \theta''_j| < \|\lambda\|_{\max}\delta_j < (m+1)\delta_j, \ j = 1, 2, \ldots, N.$$

For simplicity, we only consider the case $\theta'_j < \theta''_j$ for any $j = 1, 2, \ldots, N$. The other cases can be proved in the same way. We take an equidistant partition of $I_{\theta'',N}\backslash I_{\theta',N} = [\theta'_1, \theta''_1) \times \cdots \times [\theta'_N, \theta''_N)$, i.e.,

$$[\theta'_j, \theta''_j) = \bigcup_{l=0}^{m} [\theta_{l,j}, \theta_{l+1,j}),$$

where $\theta_{l,j} = \theta'_j + \frac{l}{m+1}(\theta''_j - \theta'_j)$, $j = 1, 2, \ldots, N$. Denote $\theta_l = (\theta_{l,1}, \ldots, \theta_{l,N})$, then $\overrightarrow{|\theta_l - \theta_{l+1}|} \in I_{\delta,N}$.

Denote $s = \min\{1, p, q\}$. We claim that

$$\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z)\right\|^s_{H^{p,q,\alpha}(U^N)} \leqslant \sum_{l=0}^{m} \left\| f(e^{i\theta_l}z) - f(e^{i\theta_{l+1}}z)\right\|^s_{H^{p,q,\alpha}(U^N)}. \qquad (3.5)$$

With this claim, we have

$$\begin{aligned}\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z)\right\|^s_{H^{p,q,\alpha}(U^N)} &\leqslant (m+1)\omega^s(\vec{\delta}, f, H^{p,q,\alpha}(U^N)) \\ &\leqslant (\|\lambda\|_{\max}+1)\omega^s(\vec{\delta}, f, H^{p,q,\alpha}(U^N)).\end{aligned}$$

Hence, by Definition 2.4,

$$\omega^s(\overrightarrow{\lambda\delta}, f, H^{p,q,\alpha}(U^N)) \leqslant (\|\lambda\|_{\max}+1)\omega^s(\vec{\delta}, f, H^{p,q,\alpha}(U^N)).$$

It remains to prove claim (3.5). We need only to consider the case $0 < p, q < \infty$, since the case of $p = \infty$ or $q = \infty$ follows from the limit process. Notice, that for $0 < r < 1$

$$\left| f(re^{i\theta'}z) - f(re^{i\theta''}z) \right| \leqslant \sum_{l=0}^{m} \left| f(re^{i\theta_{l+1}}z) - f(re^{i\theta_l}z) \right|.$$

When $1 \leqslant q < +\infty$, it follows from Minkowski's inequality that

$$M_q\left(r, f(e^{i\theta'}z) - f(e^{i\theta''}z)\right) \leqslant \sum_{l=0}^{m} M_q\left(r, f(e^{i\theta_{l+1}}z) - f(e^{i\theta_l}z)\right). \tag{3.6}$$

If $0 < q < 1$, we have

$$M_q^q\left(r, f(e^{i\theta'}z) - f(e^{i\theta''}z)\right) \leqslant \sum_{l=0}^{m} M_q^q\left(r, f(e^{i\theta_{l+1}}z) - f(e^{i\theta_l}z)\right), \tag{3.7}$$

since $(|a| + |b|)^q \leqslant |a|^q + |b|^q$ for any $(a, b) \in \mathbb{C}^2$.

We split the discussion into four cases:

(i) $1 \leqslant p < +\infty,\ 1 \leqslant q < +\infty$.
From Minkowski's inequality and (3.6), we get

$$\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z) \right\|_{H^{p,q,\alpha}(U^N)} \leqslant \sum_{l=0}^{m} \left\| f(e^{i\theta_l}z) - f(e^{i\theta_{l+1}}z) \right\|_{H^{p,q,\alpha}(U^N)}.$$

(ii) $0 < q < 1,\ 0 < q \leqslant p < +\infty$.
Since $\frac{p}{q} \geqslant 1$, from (3.7) and Minkowski's inequality we obtain

$$\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z) \right\|_{H^{p,q,\alpha}(U^N)}^q \leqslant \sum_{l=0}^{m} \left\| f(e^{i\theta_l}z) - f(e^{i\theta_{l+1}}z) \right\|_{H^{p,q,\alpha}(U^N)}^q.$$

(iii) $0 < q < 1,\ 0 < p \leqslant q \leqslant 1$.
Since $p/q \leqslant 1$, we have $(|a| + |b|)^{p/q} \leqslant |a|^{p/q} + |b|^{p/q}$, so that (3.7) implies

$$\begin{aligned} &\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z) \right\|_{H^{p,q,\alpha}(U^N)}^p \\ &\quad \leqslant \int_{I^N} \left[ \sum_{l=0}^{m} M_q^q\left(r, f(e^{i\theta_l}z) - f(e^{i\theta_{l+1}}z)\right) \right]^{\frac{p}{q}} dr_\alpha \\ &\quad \leqslant \sum_{l=0}^{m} \left\| f(e^{i\theta_l}z) - f(e^{i\theta_{l+1}}z) \right\|_{H^{p,q,\alpha}(U^N)}^p. \end{aligned}$$

(iv) $0 < p < 1,\ 1 \leqslant q < +\infty$.
Since $p/q \leqslant 1$, by (3.6) we have

$$\left\| f(e^{i\theta'}z) - f(e^{i\theta''}z) \right\|_{H^{p,q,\alpha}(U^N)}^p$$

$$\leqslant \int_{I^N} \left[ \sum_{l=0}^{m} M_q^q \left( r, f(e^{i\theta_l} z) - f(e^{i\theta_{l+1}} z) \right) \right]^{\frac{p}{q}} dr_\alpha$$
$$\leqslant \sum_{l=0}^{m} \left\| f(e^{i\theta_l} z) - f(e^{i\theta_{l+1}} z) \right\|_{H^{p,q,\alpha}(U^N)}^{p}. \qquad \square$$

In Hardy spaces $H^p(U^N)$, which consist of all $f \in H(U^N)$ such that $\sup_{r\in[0,1)^N} M_p(r, f)$ is finite, we have the following similar conclusion:

**Lemma 3.4.** *Let* $f \in H^p(U^N)$, $0 < p < +\infty$, $s = \min\{1, p\}$, *then*

$$\omega^s(\overrightarrow{\lambda\delta}, f, H^p(U^N)) \leqslant (\|\lambda\|_{\max} + 1)\omega^s(\vec{\delta}, f, H^p(U^N))$$

*for any* $\lambda = (\lambda_1, \ldots, \lambda_N) > 0,\ \delta = (\delta_1, \ldots, \delta_N) > 0.$

With the above preparation we can establish the main result of this section.

**Theorem 3.5.** *Let* $0 < p, q \leqslant +\infty,\ f \in H^{p,q,\alpha}(U^N)$, *then for any* $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$,

$$E_{\vec{k}}(f, H^{p,q,\alpha}(U^N)) \lesssim \omega\left(\overrightarrow{1/k}, f, H^{p,q,\alpha}(U^N)\right).$$

**Proof.** Let $P_m[f]$ be the polynomial in (2.1) and let $s = \min\{1, p, q\}$. We claim that

$$\|P_m[f] - f\|_{H^{p,q,\alpha}(U^N)}^s \leqslant \int_{[-\pi,\pi)^N} \left\| f(e^{i\varphi} z) - f(z) \right\|_{H^{p,q,\alpha}(U^N)}^s dv_m^{\rho,s,a}(\varphi). \quad (3.8)$$

We need only to prove (3.8) for $0 < p, q < \infty$, since the remaining cases can be proved by suitable modification. Set $\eta = s$ in (3.2), take the $q/s$th power, then integrate over $T^N$, and apply Fubini's theorem to yield

$$M_q^s(r, P_m[f] - f) \leqslant \int_{[-\pi,\pi)^N} M_q^s \left( r, f(e^{i\varphi} z) - f(z) \right) dv_m^{\rho,s,a}(\varphi).$$

The claim then follows from Minkowski's inequality.

From (3.8), we have

$$\|P_m[f] - f\|_{H^{p,q,\alpha}(U^N)}^s \leqslant \int_{[-\pi,\pi)^N} \omega^s \left( \bar{\varphi}, f, H^{p,q,\alpha}(U^N) \right) dv_m^{\rho,s,a}(\varphi),$$

where $\bar{\varphi}$ denotes the vector $(|\varphi_1|, \ldots, |\varphi_N|)$.

Pick $a > 0$ such that $s(a + 1) > 2$. By Lemmas 3.3 and 3.2 we obtain

$$\|P_m[f] - f\|_{H^{p,q,\alpha}(U^N)}^s$$
$$\leqslant \omega^s \left( \overrightarrow{1/m}, f, H^{p,q,\alpha}(U^N) \right) \int_{[-\pi,\pi)^N} (\|m\varphi\|_{\max} + 1)\, dv_m^{\rho,s,a}(\varphi)$$
$$\lesssim \omega^s \left( \overrightarrow{1/m}, f, H^{p,q,\alpha}(U^N) \right). \quad (3.9)$$

Applying Lemmas 2.6, 3.3, and 2.3, we finally get

$$E_{\vec{k}}(f, H^{p,q,\alpha}(U^N)) \lesssim \omega\left(\overrightarrow{1/k}, f, H^{p,q,\alpha}(U^N)\right). \qquad \square$$

## 4. Sobolev space

Let $0 < p, q \leqslant +\infty$, $\alpha > 0$, $m \in \mathbb{Z}_+$. The function $f \in H(U^N)$ is said to belong to the *Sobolev space* $W^m_{p,q,\alpha}(U^N)$ if $D^\beta f \in H^{p,q,\alpha}(U^N)$ for any $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{Z}^N_+$ with $|\beta| \leqslant m$, where

$$D^\beta = \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \cdots \partial z_N^{\beta_N}}.$$

For the Sobolev spaces, we have the following Jackson's theorem:

**Theorem 4.1.** *Let $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and let $m$ be a non-negative integer. If $f \in W^m_{p,q,\alpha}(U^N)$, then for any $k \in \mathbb{N}$*

$$E_k(f, H^{p,q,\alpha}(U^N)) \lesssim \sum_{|\beta|=m} \frac{1}{k^m} \cdot \omega\left(\frac{1}{k}, D^\beta f, H^{p,q,\alpha}(U^N)\right).$$

We need some lemmas before proving Theorem 4.1.

**Lemma 4.2.** *Let $g \in H(U^N)$ and $0 < p, q \leqslant \infty$, then for any $\delta \in (0, \infty)^N$*

$$\omega(\vec{\delta}, g, H^{p,q,\alpha}(U^N)) \lesssim \sum_{j=1}^N \delta_j \left\| \frac{\partial g}{\partial z_j} \right\|_{H^{p,q,\alpha}(U^N)}.$$

**Proof.** Let $z \in U^N$ and write $z = r\zeta$ with $r \in [0,1)^N$ and $\zeta \in T^N$. We fix $h = (h_1, \ldots, h_N) \in I_{\delta,N}$ and denote

$$\gamma_j = \left(r_1\zeta_1, \ldots, r_{j-1}\zeta_{j-1}\right), \quad \mu_j = \left(r_{j+1}e^{ih_{j+1}}\zeta_{j+1}, \ldots, r_N e^{ih_N}\zeta_N\right).$$

Consequently,

$$\begin{aligned}
|g(e^{ih}z) - g(z)| &= |g(re^{ih}\zeta) - g(r\zeta)| \\
&\leqslant \sum_{j=1}^N |g(\gamma_j, r_j e^{ih_j}\zeta_j, \mu_j) - g(\gamma_j, r_j\zeta_j, \mu_j)| \\
&= \sum_{j=1}^N \left| \int_0^{h_j} \frac{\partial g}{\partial \theta_j}(\gamma_j, r_j e^{i\theta_j}\zeta_j, \mu_j)\, d\theta_j \right|.
\end{aligned}$$

If $1 \leqslant q < \infty$, then Minkowski's inequality shows

$$
\begin{aligned}
& M_q\left(r, g(e^{ih}z) - g(z)\right) \\
&\quad = \left( \int_{T^N} \left| g(re^{ih}\zeta) - g(r\zeta) \right|^q \, d\sigma_N(\zeta) \right)^{1/q} \\
&\quad \lesssim \sum_{j=1}^{N} \left( \int_{T^N} \left| \int_0^{h_j} \frac{\partial g}{\partial \theta_j}(\gamma_j, r_j e^{i\theta_j}\zeta_j, \mu_j)\, d\theta_j \right|^q \, d\sigma_N(\zeta) \right)^{1/q} \\
&\quad \lesssim \sum_{j=1}^{N} \int_0^{h_j} \left( \int_{T^N} \left| \frac{\partial g}{\partial \theta_j}(\gamma_j, r_j e^{i\theta_j}\zeta_j, \mu_j) \right|^q \, d\sigma_N(\zeta) \right)^{1/q} \, d\theta_j .
\end{aligned}
$$

Therefore,

$$
M_q\left(r, g(e^{ih}z) - g(z)\right) \lesssim \sum_{j=1}^{N} \delta_j M_q\left(r, \frac{\partial g}{\partial z_j}\right).
$$

For $q = +\infty$,

$$
\begin{aligned}
& M_\infty\left(r, g(e^{i\theta}z) - g(z)\right) \\
&\quad \leqslant \sup_{\zeta \in T^N} \sum_{j=1}^{N} \int_0^{h_j} \left| \frac{\partial}{\partial z_j} g\left(\gamma_j, r_j e^{i\theta_j}\zeta_j, \mu_j\right) \right| d\theta \\
&\quad \leqslant \sup_{\zeta \in T^N} \sum_{j=1}^{N} h_j \left| \frac{\partial g}{\partial z_j}(re^{i\theta_0}\zeta) \right| \\
&\quad \leqslant \sum_{j=1}^{N} \delta_j M_\infty\left(r, \frac{\partial g}{\partial z_j}\right).
\end{aligned}
$$

Moreover, in case of $1 \leqslant q \leqslant +\infty$, $0 < p < +\infty$

$$
M_q^p\left(r, g(e^{ih}z) - g(z)\right) \lesssim \sum_{j=1}^{N} \delta_j^p M_q^p\left(r, \frac{\partial g}{\partial z_j}\right),
$$

so that

$$
\left\| g(e^{ih}z) - g(z) \right\|_{H^{p,q,\alpha}(U^N)} \lesssim \sum_{j=1}^{N} \delta_j \left\| \frac{\partial g}{\partial z_j} \right\|_{H^{p,q,\alpha}(U^N)},
$$

which implies

$$
\omega(\vec{\delta}, g, H^{p,q,\alpha}(U^N)) \lesssim \sum_{j=1}^{N} \delta_j \left\| \frac{\partial g}{\partial z_j} \right\|_{H^{p,q,\alpha}(U^N)}.
$$

It is clear that the inequality also holds true in case of $p = +\infty$. □

**Corollary 4.3.** *Let $g \in H(U^N)$, $0<p\leqslant+\infty$, and $1\leqslant q\leqslant+\infty$, then for any $0<\delta<+\infty$*

$$\omega(\delta, g, H^{p,q,\alpha}(U^N)) \lesssim \delta \sum_{j=1}^{N} \left\| \frac{\partial g}{\partial z_j} \right\|_{H^{p,q,\alpha}(U^N)}.$$

**Lemma 4.4.** *Suppose $\mathcal{Q}_j \in H(U^N)$, $j=1,2,\ldots,N$, then there exists $P \in H(U^N)$ such that*

$$\frac{\partial P}{\partial z_j}(z) = \mathcal{Q}_j(z), \quad j=1,2,\ldots,N, \tag{4.1}$$

*if and only if*

$$\frac{\partial \mathcal{Q}_j}{\partial z_i}(z) = \frac{\partial \mathcal{Q}_i}{\partial z_j}(z), \quad i,j=1,2,\ldots,N. \tag{4.2}$$

*When* (4.2) *holds, the solutions are given by*

$$\begin{aligned} P(z) = & \int_0^{z_1} \mathcal{Q}_1(\zeta_1, z_2, \ldots, z_N)\, d\zeta_1 + \int_0^{z_2} \mathcal{Q}_2(0, \zeta_2, z_3, \ldots, z_N)\, d\zeta_2 \\ & + \cdots + \int_0^{z_N} \mathcal{Q}_N(0, \ldots, 0, \zeta_N)\, d\zeta_N + C, \end{aligned} \tag{4.3}$$

*where $C$ is a constant.*

**Proof.** We only need to verify the sufficiency. Namely, for any given $\mathcal{Q}_j \in H(U^N)$ satisfying (4.2), if we take $P$ as in (4.3) then $P$ is a solution of Eq. (4.1). Indeed,

$$\begin{aligned} \frac{\partial P}{\partial z_j}(z) = & \int_0^{z_1} \frac{\partial}{\partial z_j} \mathcal{Q}_1(\zeta_1, z_2, \ldots, z_N)\, d\zeta_1 + \int_0^{z_2} \frac{\partial}{\partial z_j} \mathcal{Q}_2(0, \zeta_2, z_3, \ldots, z_N)\, d\zeta_2 \\ & + \cdots + \frac{\partial}{\partial z_j} \int_0^{z_j} \mathcal{Q}_j(0, \ldots, 0, \zeta_j, z_{j+1}, \ldots, z_N)\, d\zeta_j \\ = & \int_0^{z_1} \frac{\partial}{\partial z_1} \mathcal{Q}_j(\zeta_1, z_2, \ldots, z_N)\, d\zeta_1 + \int_0^{z_2} \frac{\partial}{\partial z_2} \mathcal{Q}_j(0, \zeta_2, z_3, \ldots, z_N)\, d\zeta_2 \\ & + \cdots + \mathcal{Q}_j(0, \ldots, 0, z_j, z_{j+1}, \ldots, z_N) \\ = & \mathcal{Q}_j(z_1, \ldots, z_N) - \mathcal{Q}_j(0, z_2, \ldots, z_N) + \mathcal{Q}_j(0, z_2, \ldots, z_N) \\ & + \cdots - \mathcal{Q}_j(0, \ldots, 0, z_j, \ldots, z_N) + \mathcal{Q}_j(0, \ldots, 0, z_j, \ldots, z_N) \\ = & \mathcal{Q}_j(z). \qquad \square \end{aligned}$$

In Sobolev space, we estimate the error of the polynomial approximation by the slice modulus of smoothness. To this end, we need to modify the approximation polynomial given by (2.1). Namely, we define

$$F_k[f](z) \stackrel{\text{def}}{=} \int_{[-\pi,\pi)} f(\rho e^{i\varphi} z)\, d\mu_\rho(\varphi)$$

for any $0 < \rho < 1,\ k \in \mathbb{N},\ z \in U^N$. Here

$$d\mu_\rho(\varphi) = (A_k^a)^{-1}(\rho e^{i\varphi})^{1-k}\left[\frac{1-(\rho e^{i\varphi})^{k+1}}{1-\rho e^{i\varphi}}\right]^{a+1} d\varphi$$

being the normalized Lebesgue measure on $[-\pi, \pi)^N$ for any fixed $a > 0$.

**Lemma 4.5.** *Let* $f \in H(U^N)$, *then*

(A) $F_k[f](z)$ *is a polynomial of degree at most* $k-1$; $F_k[1] \equiv 1$.
(B) *For any* $0 < p, q < \infty$ *and* $\alpha > 0$ *we have*

$$\|F_k[f] - f\|_{H^{p,q,\alpha}(U^N)} \lesssim \omega\left(\frac{1}{k}, f, H^{p,q,\alpha}(U^N)\right).$$

(C) *For any* $1 \leqslant i, j \leqslant N$

$$\frac{\partial}{\partial z_i} F_k\left[\frac{\partial f}{\partial z_j}\right](z) = \frac{\partial}{\partial z_j} F_k\left[\frac{\partial f}{\partial z_i}\right](z).$$

**Proof.** (A) follows from Lemma 2.3. (B) follows from the same ideas as in the proof of Theorem 3.5. (C) follows from the integral representation of $F_k$. □

Now we can give the proof of the main result in this section.

**Proof of Theorem 4.1.** Let $m$ and $k$ be integers and $k \geqslant m+1$. For the proof of the theorem it is sufficient to construct linear operators $P_{m,k}$ with the following properties: For any $f \in H(U^N)$, $P_{m,k}[f](z)$ is a polynomial of degree at most $k$ satisfying

$$\frac{\partial}{\partial z_i} P_{m,k}\left[\frac{\partial f}{\partial z_j}\right](z) = \frac{\partial}{\partial z_j} P_{m,k}\left[\frac{\partial f}{\partial z_i}\right](z) \tag{4.4}$$

for any $i, j = 1, 2, \ldots, N$, and, furthermore, if $f \in W^m_{p,q,\alpha}(U^N)$ then

$$\left\| P_{m,k}[f] - f \right\|_{H^{p,q,\alpha}(U^N)} \lesssim \sum_{|\beta|=m} \frac{1}{k^m}\omega\left(\frac{1}{k}, D^\beta f, H^{p,q,\alpha}(U^N)\right). \tag{4.5}$$

To construct $P_{m,k}$ we apply induction on $m$. When $m = 0$ we set $P_{0,k} = F_k$. In this case the properties follow from Lemma 4.5. Assume now that the properties hold true for $m = l \geqslant 0$.

For $m = l+1$ and $f \in W^m_{p,q,\alpha}(U^N)$, we have $D^\beta f \in H^{p,q,\alpha}(U^N)$, $|\beta| = l+1$. Applying the inductive assumption to $\dfrac{\partial f}{\partial z_j}$ $(j = 1, 2, \ldots, N)$, then for any $k \geqslant m = l+1$

we get

$$\left\| P_{l,k-1}\left[\frac{\partial f}{\partial z_j}\right] - \frac{\partial f}{\partial z_j}\right\|_{H^{p,q,\alpha}(U^N)} \lesssim \sum_{|\beta|=l} \frac{1}{(k-1)^l} \cdot \omega\left(\frac{1}{k-1}, \frac{\partial^{|\beta|+1}}{\partial z^\beta \partial z_j} f, H^{p,q,\alpha}(U^N)\right) \tag{4.6}$$

and

$$\frac{\partial}{\partial z_i} P_{l,k-1}\left[\frac{\partial f}{\partial z_j}\right] = \frac{\partial}{\partial z_j} P_{l,k-1}\left[\frac{\partial f}{\partial z_i}\right] \quad \forall\, i, j = 1, 2, \ldots, N.$$

Motivated by Lemma 4.4, we introduce the operator:

$$T(\mathcal{Q}_1, \ldots, \mathcal{Q}_N)(z) = \int_0^{z_1} \mathcal{Q}_1(\zeta_1, z_2, \ldots, z_N)\, d\zeta_1 + \int_0^{z_2} \mathcal{Q}_2(0, \zeta_2, z_3, \ldots, z_N)\, d\zeta_2 + \cdots + \int_0^{z_N} \mathcal{Q}_N(0, \ldots, 0, \zeta_N)\, d\zeta_N.$$

If we set

$$P^*_{l+1,k}[f](z) = T\left(P_{l,k-1}\left[\frac{\partial f}{\partial z_1}\right], \ldots, P_{l,k-1}\left[\frac{\partial f}{\partial z_N}\right]\right)(z)$$

then, by construction, $P^*_{l+1,k}[f](z)$ is a polynomial of degree at most $k$, and Lemma 4.4 tells us that

$$\frac{\partial}{\partial z_j} P^*_{l+1,k}[f](z) = P_{l,k-1}\left[\frac{\partial f}{\partial z_j}\right](z),$$

which implies

$$\frac{\partial}{\partial z_j} P^*_{l+1,k}\left[\frac{\partial f}{\partial z_i}\right](z) = P_{l,k-1}\left[\frac{\partial^2 f}{\partial z_i \partial z_j}\right](z) = \frac{\partial}{\partial z_i} P^*_{l+1,k}\left[\frac{\partial f}{\partial z_j}\right](z).$$

On the other hand, by (4.6) and Lemma 3.3 we have

$$\begin{aligned} &\left\| \frac{\partial}{\partial z_j} P^*_{l+1,k}[f] - \frac{\partial f}{\partial z_j}\right\|_{H^{p,q,\alpha}(U^N)} \\ &\quad = \left\| P_{l,k-1}\left[\frac{\partial f}{\partial z_j}\right] - \frac{\partial f}{\partial z_j}\right\|_{H^{p,q,\alpha}(U^N)} \\ &\quad \lesssim \sum_{|\beta|=l} \frac{1}{(k-1)^l} \cdot \omega\left(\frac{1}{k-1}, \frac{\partial^{|\beta|+1}}{\partial z^\beta \partial z_j} f, H^{p,q,\alpha}(U^N)\right) \\ &\quad \lesssim \sum_{|\beta|=l} \frac{1}{k^l} \cdot \omega\left(\frac{1}{k}, \frac{\partial^{|\beta|+1}}{\partial z^\beta \partial z_j} f, H^{p,q,\alpha}(U^N)\right), \end{aligned}$$

so that Lemma 4.2 shows

$$\begin{aligned}
&\omega\left(\frac{1}{k}, P^*_{l+1,k}[f]-f, H^{p,q,\alpha}(U^N)\right)\\
&\lesssim \sum_{j=1}^{N}\frac{1}{k}\left\|\frac{\partial}{\partial z_j}P^*_{l+1,k}[f]-\frac{\partial f}{\partial z_j}\right\|_{H^{p,q,\alpha}(U^N)}\\
&\lesssim \sum_{|\beta|=l+1}\frac{1}{k^{l+1}}\cdot\omega\left(\frac{1}{k}, D^\beta f, H^{p,q,\alpha}(U^N)\right).
\end{aligned}\tag{4.7}$$

Thus, the same argument as in (3.9) shows

$$||P_{0,k}f-f||_{H^{p,q,\alpha}(U^N)}\lesssim\omega\left(\frac{1}{k}, f, H^{p,q,\alpha}(U^N)\right).$$

Now, replacing $f$ with $P^*_{l+1,k}[f]-f$ and applying (4.7) yields

$$\begin{aligned}
&\left\|P_{0,k}\left[P^*_{l+1,k}[f]-f\right]-\left(P^*_{l+1,k}[f]-f\right)\right\|_{H^{p,q,\alpha}(U^N)}\\
&\lesssim\omega\left(\frac{1}{k}, P^*_{l+1,k}[f]-f, H^{p,q,\alpha}(U^N)\right)\\
&\lesssim \sum_{|\beta|=l+1}\frac{1}{k^{l+1}}\cdot\omega\left(\frac{1}{k}, D^\beta f, H^{p,q,\alpha}(U^N)\right).
\end{aligned}$$

Motivated by this inequality, we can define

$$P_{l+1,k}[f](z)=P^*_{l+1,k}[f](z)+P_{0,k}[f](z)-P_{0,k}[P^*_{l+1,k}[f]](z),$$

so that the preceding inequality shows exactly that (4.5) holds true. Clearly, $P_{l+1,k}[f](z)$ is a polynomial of degree at most $k$, and $P_{l+1,k}[f](z)$ satisfies (4.4).

From Lemmas 2.7 and 3.4, we get the desired result. □

## 5. Hardy spaces

As in Bergman-type spaces and Sobolev spaces, we can establish similar conclusions in Hardy spaces $H^p(U^N)$.

**Theorem 5.1.** *Let* $0<p\leqslant+\infty$, $f\in H^p(U^N)$, *then for any* $k\in\mathbb{N}^N$,

$$E_{\vec{k}}(f, H^p(U^N))\lesssim\omega\left(\overrightarrow{1/k}, f, H^p(U^N)\right).$$

**Proof.** Denote $s=\min\{1,p\}$ and set $\eta=s$ in (3.2), then for any $m\in\mathbb{N}^N$

$$|P_m[f](z)-f(z)|^s\lesssim\int_{[-\pi,\pi)^N}\left|f(e^{i\varphi}z)-f(z)\right|^s\,dv_m^{\rho,s,a}(\varphi).$$

By Minkowski's inequality

$$
\begin{aligned}
&\|P_m[f](z) - f(z)\|^s_{H^p(U^N)} \\
&= \lim_{r\to 1^-} M_p^s\,(r, P_m[f] - f) \\
&\lesssim \lim_{r\to 1^-} \int\limits_{[-\pi,\pi)^N} M_p^s\left(r, f(e^{i\varphi}z) - f(z)\right) d\nu_m^{\rho,s,a}(\varphi) \\
&= \int\limits_{[-\pi,\pi)^N} \left\| f(e^{i\varphi}z) - f(z)\right\|^s_{H^p(U^N)} d\nu_m^{\rho,s,a}(\varphi) \\
&\leqslant \int\limits_{[-\pi,\pi)^N} \omega^s\left(\bar{\varphi}, f, H^p(U^N)\right) d\nu_m^{\rho,s,a}(\varphi).
\end{aligned}
$$

Here we used the monotone convergence theorem. By Lemmas 3.2 and 3.4, the above item can be further estimated by

$$
\begin{aligned}
&\omega^s\left(\overrightarrow{1/m}, f, H^P(U^N)\right) \int\limits_{[-\pi,\pi)^N} (\|m\varphi\|_{\max} + 1)\, d\nu_m^{\rho,s,a}(\varphi) \\
&\lesssim \omega^s\left(\overrightarrow{1/m}, f, H^P(U^N)\right).
\end{aligned}
$$

The result then follows from Lemmas 2.6 and 3.4. □

**Definition 5.2.** Let $0 < p \leqslant +\infty$, $\alpha > 0$, and $m$ be a non-negative integer. A function $f \in H(U^N)$ is said to belong to Hardy–Sobolev space $W_m^p(U^N)$ if $D^\beta f \in H^p(U^N)$ for any multi-index $\beta$ with $|\beta| \leqslant m$.

A similar argument as in Theorem 4.1 gives the following result:

**Theorem 5.3.** *Let $1 \leqslant p \leqslant +\infty$ and $m$ be a non-negative integer. If $f \in W_m^p(U^N)$ then*

$$
E_k(f, H^p(U^N)) \lesssim \sum_{|\beta|=m} \frac{1}{k^m} \cdot \omega\left(\frac{1}{k}, D^\beta f, H^p(U^N)\right).
$$

This extends the result of Colzani [3] for $m = 0$.

## 6. Polydisc algebra

The *polydisc algebra* $A(U^N)$ is the class of all holomorphic functions in $U^N$ with continuous extension to the closure of $U^N$. Recalling the notation $B_{k,\beta}$ in Lemma 2.1 we define

$$I_k[f](z) \overset{\text{def}}{=} \frac{1}{B_{k,\beta}} \int_{-\pi}^{\pi} \sum_{u=1}^{l} (-1)^{1+u} \binom{l}{u} f(e^{iu\varphi} z) T_k^{\beta}(\varphi)\, d\,\varphi$$

for any fixed $l, \beta \in \mathbb{N}$.

**Lemma 6.1.** *Let $f \in A(U^N)$ and $k \in \mathbb{N}$, then $I_k[f](z)$ is a polynomial of degree at most $\beta(k-1)$.*

**Proof.** Fix $z \in T^N$. From the homogeneous expansion $f(z) = \sum_{m=0}^{\infty} P_m(z)$, Lemma 2.1 implies that for $u = 1, 2, \ldots, l$

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(e^{iu\varphi} z) T_{k,\beta}(\varphi)\, d\varphi \\
&\quad = \int_{-\pi}^{\pi} \sum_{m=0}^{\infty} P_m(z) e^{imu\varphi} \cdot \sum_{l=0}^{\beta(k-1)} C_l \cos l\varphi\, d\varphi \\
&\quad = \sum_{mu \leqslant \beta(k-1)} C_{mu} P_m(z) \int_{-\pi}^{\pi} e^{imu\varphi} \cos mu\varphi\, d\varphi.
\end{aligned}$$

This shows $I_k[f](z)$ is a polynomial of degree at most $\beta(k-1)$. □

**Lemma 6.2** (*DeVore et al. [4]*). *Suppose* $0 < \delta, \lambda < +\infty,\ f \in A(U)$, *then*

$$\omega_l\left(\lambda\delta, f, A(U)\right) \leqslant (\lambda+1)^l \omega_l(\delta, f, A(U)).$$

**Lemma 6.3.** *Suppose* $0 < \delta, \lambda < +\infty,\ f \in A(U^N)$, *then*

$$\omega_l\left(\lambda\delta, f, A(U^N)\right) \leqslant (\lambda+1)^l \omega_l(\delta, f, A(U^N)).$$

**Proof.** For any $\zeta \in T^N$, consider the slice function $f_\zeta$ of $f$, where $f_\zeta(\mu) = f(\mu\zeta),\ \mu \in U$. Then Lemma 6.2 implies

$$\omega_l(\lambda\delta, f_\zeta, A(U)) \leqslant (\lambda+1)^l \omega_l(\delta, f_\zeta, A(U)).$$

By Definition 2.5 we have

$$\omega_l(\delta, f_\zeta, A(U)) \leqslant \omega_l(\delta, f, A(U^N))$$

for arbitrary $\zeta \in T^N$, so that

$$\omega_l(\lambda\delta, f_\zeta, A(U)) \leqslant (\lambda+1)^l \omega_l(\delta, f, A(U^N)).$$

Again from Definition 2.5 we have

$$\omega_l(\lambda\delta, f, A(U^N)) \leqslant (\lambda+1)^l \omega_l(\delta, f, A(U^N)). \qquad \square$$

The following theorem is our main result in $A(U^N)$, which recovers Theorem 1.2 when $N = 1$ and $l = 1$.

**Theorem 6.4.** *Let $f \in A(U^N)$, then for any $k, l \in \mathbb{N}$,*

$$E_k(f, A(U^N)) \overset{\text{def}}{=} \inf_{P_k \in \mathrm{M}_k} \max_{z \in \overline{U^N}} |f(z) - P_k(z)| \lesssim \omega_l \left( \frac{1}{k}, f, A(U^N) \right).$$

**Proof.** For any $\zeta \in T^N$, from the definition of $\Delta^l_{2\varphi} f(\zeta)$ and Lemma 6.3 we get

$$\begin{aligned}
\left| I_k[f](\zeta) - f(\zeta) \right| &= \frac{2}{B_{k,\beta}} \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-1)^{l+1} \Delta^l_{2\varphi} f(\zeta) T_k^\beta(2\varphi)\, d\varphi \right| \\
&\leqslant \frac{2}{B_{k,\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \omega_l(2|\varphi|, f, A(U^N)) T_k^\beta(2\varphi)\, d\varphi \\
&\leqslant \frac{4}{B_{k,\beta}} \omega_l \left( \frac{1}{k}, f, A(U^N) \right) \int_0^{\frac{\pi}{2}} (1 + 2k\varphi)^l T_k^\beta(2\varphi)\, d\varphi \\
&= \frac{2}{B_{k,\beta}} \omega_l \left( \frac{1}{k}, f, A(U^N) \right) \int_0^{\pi} \sum_{u=0}^{l} \binom{l}{u} (k\theta)^u T_k^\beta(\theta)\, d\theta.
\end{aligned}$$

Setting $2\beta > l + 1$ and applying Lemma 2.1 we find

$$\left| I_k[f](\zeta) - f(\zeta) \right| \leqslant C_\beta \cdot \omega_l \left( \frac{1}{k}, f, A(U^N) \right).$$

The result now follows from Lemmas 2.7 and 6.3. $\square$

## 7. Lipschitz spaces

The *Lipschitz space* $Lip_\gamma(U^N)$, $0 < \gamma \leqslant 1$, consists of all holomorphic functions $f \in A(U^N)$ satisfying

$$|\Delta_h f(\zeta)| = |f(e^{ih}\zeta) - f(\zeta)| \leqslant L|h|^\gamma$$

for any $\zeta \in T^N$ and $h \in \mathbb{R}$. Here $L > 0$ being the so-called Lipschitz constant.

**Definition 7.1.** Let $m \in \mathbb{Z}_+$. The function $f \in A(U^N)$ is said to belong to the Lipschitz space $Lip^m_\gamma(U^N)$ if $D^\alpha f \in Lip_\gamma(U^N)$ for any $|\alpha| \leqslant m$.

**Theorem 7.2.** *If $f \in Lip^m_\gamma(U^N)$ then*

$$E_k(f, A(U^N)) \leqslant L \frac{C_{m,\gamma}}{k^{m+\gamma}}, \quad k \in \mathbb{N}.$$

**Proof.** By induction, one can easily verify that for any $f \in A(U^N)$

$$\begin{aligned}
&\Delta_h^{m+1} f(e^{i\theta_0}\zeta) \\
&\quad = \int_{[0,h]^{m+1}} \frac{d^{m+1} f}{d\theta^{m+1}} \left(e^{i(\theta_0+r_1+\cdots+r_m+r_{m+1})}\zeta\right) dr_{m+1}\, dr \\
&\quad = \int_{[0,h]^m} \frac{d^m f}{d\theta^m} \left(e^{i(\theta_0+h+r_1+\cdots+r_m)}\zeta\right) - \frac{d^m f}{d\theta^m} \left(e^{i(\theta_0+r_1+\cdots+r_m)}\zeta\right) dr. \qquad (7.1)
\end{aligned}$$

If $f \in Lip_\gamma^m(U^N)$ then $D^\alpha f \in Lip_\gamma(U^N)$, $|\alpha| \leqslant m$, so that

$$|D^\alpha f(e^{ih}\zeta) - D^\alpha f(\zeta)| \leqslant L|h|^\gamma.$$

Since

$$\frac{df}{d\theta}(e^{i\theta}\zeta) = i \sum_{j=1}^N \frac{\partial f}{\partial z_j}(e^{i\theta}\zeta) e^{i\theta}\zeta_j,$$

we have

$$\begin{aligned}
\left|\frac{df}{d\theta}(e^{ih}\zeta) - \frac{df}{d\theta}(\zeta)\right| &\leqslant \sum_{j=1}^N \left|\frac{\partial f}{\partial z_j}(e^{ih}\zeta)e^{ih} - \frac{\partial f}{\partial z_j}(\zeta)\right| \\
&\leqslant \sum_{j=1}^N \left|\frac{\partial f}{\partial z_j}(e^{ih}\zeta)\right| \left|e^{ih} - 1\right| + \sum_{j=1}^N \left|\frac{\partial f}{\partial z_j}(e^{ih}\zeta) - \frac{\partial f}{\partial z_j}(\zeta)\right| \\
&= O(h^\gamma)
\end{aligned}$$

for $0 < \gamma \leqslant 1$, namely, $\frac{\partial f}{\partial \theta}(e^{i\theta}\zeta)$, as a function of $\theta$, is a Lipschitz function of degree $\gamma$. By induction, we know that $\frac{\partial^m f}{\partial \theta^m}(e^{i\theta}\zeta)$, as a function of $\theta$, is a Lipschitz function of degree $\gamma$.

Therefore, formula (7.1) implies

$$\left|\Delta_h^{m+1} f(\zeta)\right| \leqslant L|h|^{m+\gamma}.$$

Hence

$$\omega_{m+1}\left(\frac{1}{k}, f, A(U^N)\right) \leqslant \frac{L}{k^{m+\gamma}}.$$

The result now follows from Theorem 6.4. □

When $N = 1$ and $m = 0$, Theorem 7.2 recovers Theorem 1.3.

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